Error Analysis of Euler Angle Transformations

RICHARD L. PIO*
Hughes Aircraft Company, Canoga Park, Calif.

Error analyses of Euler angle transformations arise in the design of precision pointing systems, guidance systems and other systems containing gimbals. The generalized problem, including nonorthogonality of nominally orthogonal coordinate axes as well as errors in the Euler angles, is treated. The usual approach by tedious matrix techniques is simplified to yield a vector solution by application of a similarity transformation to the skew-symmetric error matrices. A vector solution is obtained directly by invoking the vector property of infinitesimal rotations. Piograms, symbolic representations of Euler angle transformations, are used to formulate the problem and to develop the vector solution.

Introduction

SEQUENCES of angular rotations are universally used in the analysis of rigid-body dynamics. Examples can be widely found in analysis of precision pointing systems and aircraft, missiles, and space vehicles. As shown by Euler, a minimum of three rotations are required to specify the relative orientation of two orthogonal coordinate systems with arbitrary attitudes. Any physical realization of coordinate transformations, as with gimbals, for example, will introduce angular errors so that each coordinate axis is perturbed from its idealized position. The most general error analysis of an Euler angle sequence must include the effects of nonorthogonality of the nominally orthogonal coordinate axes as well as errors in the angles of rotation.

An example problem is presented to illustrate the various approaches available for error analysis of an Euler angle sequence. The usual but tedious solution by matrix techniques is given, A vector solution is obtained by applying a similarity transformation to the matrix solution. In a second approach a vector solution to the problem is obtained directly by invoking the vector property of infinitesimal rotations. Piograms, symbolic representations of rotational coordinate transformations,[1,2] are used to define the problem and are shown to give the vector solution in a particularly compact way.

Three different Euler angle sequences are commonly used in the literature. One of these three is used as the example problem in this paper. The method of solution presented in the paper is applicable to any of the twelve possible three-angle Euler sequences. The method is applicable to rotational sequences of any length by an obvious extension of the technique.

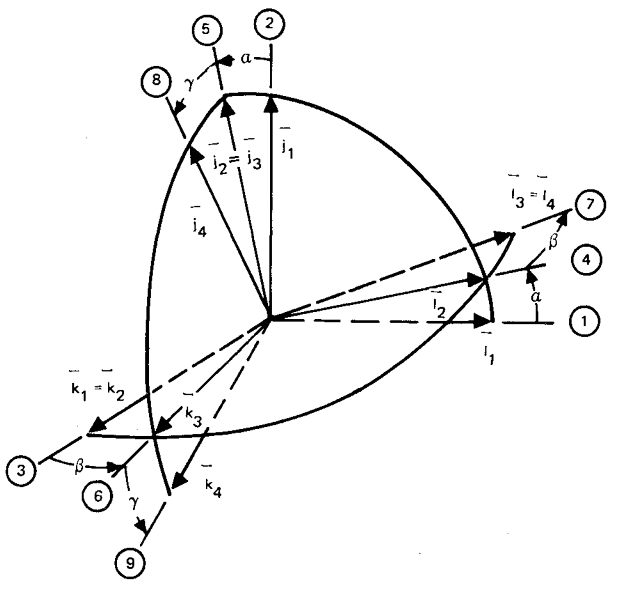

Fig. 1 Definition of coordinate systems for sample problem.

Presented as Paper 72-851 at the AIAA Guidance and Control Conference, Stanford, Calif., August 14–16, 1972; submitted August 24, 1972; revision received December 11, 1972.

Index categories: Spacecraft Attitude Dynamics and Control; Navigation, Control, and Guidance Theory.

* Manager, Avionic Systems Department, Advanced Programs Division.

Problem Definition

The three-angle sequence shown in Fig. 1 is used as the basis of the sample problem to illustrate methods for performing an error analysis of an Euler angle transformation. Coordinate system 1 (defined by an orthogonal set of unit vectors $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$) is a right-handed orthogonal reference system. The orientation of coordinate system 4 (orthogonal unit vectors $\mathbf{i}_4, \mathbf{j}_4, \mathbf{k}_4$) relative to coordinate system 1 is specified by a set of Euler angles α, β, γ defined as rotations about unit vectors $\mathbf{k}_1, \mathbf{j}_2, \mathbf{i}_3$, respectively. Coordinate systems 2 and 3 are intermediate orthogonal sets. The Piogram equivalent to Fig. 1 is shown in Fig. 2.

The basic characteristics of the Piogram are listed below.

a) The characteristics and sign conventions for the symbol used to represent a single rotation of coordinate axes are shown in Fig. 3.

b) Small angles are represented as shown in Fig. 3c. Second-order effects are discarded by restricting each path through the diagram to traverse only one small angle.

c) All coordinate systems are right-handed and positive rotations are defined by the right-hand rule.

d) A sequence of rotations is represented by joining end-to-end

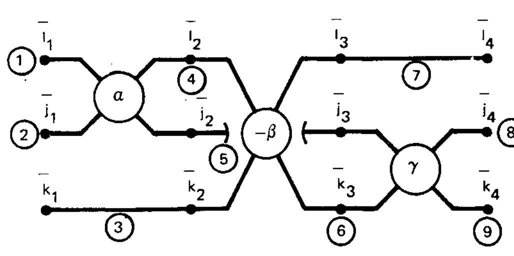

Fig. 2 Piogram equivalent to Fig. 1.

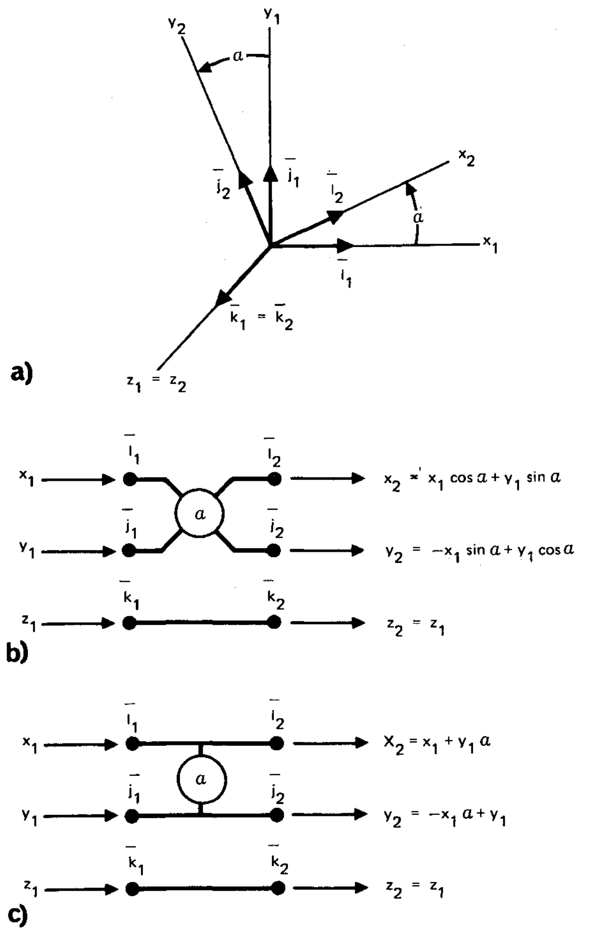

Fig. 3 Properties of Piograms: a) Rotation of coordinate axes about $\mathbf{k}_1$; b) Piogram representation of (a); c) Small-angle approximation to (b).

the symbolic representations (Piograms) of the individual rotations. See Fig. 2.

e) Positive rotations about x or z axes are shown with a plus sign. Positive rotations about the y axis are shown with a negative sign. See Fig. 2.

f) The Piogram for an inverse transformation is formed by reversing the Piogram for the forward transformation and changing the sign of each of the angles.

g) The three dots in a vertical array between each of the individual rotations always correspond to an orthogonal set of unit vectors.

h) Those unit vectors that are coincident or parallel when the rotation angles are zero lie on the same horizontal line in the Piogram.

i) Scalar products of vectors are determined from the Piogram by tracing the unique paths between the points representing the vectors in the diagram.

j) The number of unique paths—the same as the number of terms in the scalar product—is a member of the Fibonacci sequence defined by

$N_n = N_{n-1} + N_{n-2}$

$N_1 = 1, N_2 = 1$

N_k = number of unique paths between two unit vectors when k rotations lie between the vectors.

Any physical realization of the coordinate transformations in Figs. 1 or 2 will involve nonorthogonality of the basic coordinate axes as well as errors in the magnitude of the angles of rotation. The net effect of these errors is to cause coordinate system 4 to be perturbed to a new orientation defined by an orthogonal set of unit vectors $\mathbf{i}_4'$, $\mathbf{j}_4'$, $\mathbf{k}_4'$. The error analysis problem posed in this example is that of determining the orientation of orthogonal coordinate system 4′ in terms of α, β, γ and the errors committed in performing the rotations.

Attention is restricted here to the case where the angular errors are small. This is a useful case in itself because there are many precision systems for which angular errors must be held to very small values. The small-angle approximation also provides an easily obtained limiting case for large errors.

The orientation of coordinate system 4′, the perturbed system, relative to the original coordinate system 4, can be specified by a set of three angles, ε_1, ε_2, ε_3, since it is always true that two orthogonal coordinate systems can be related by three angles (Fig. 4). For small angular errors, the ε_i's will also be small. (In Fig. 4 a Piogram for small angles is used implying the assumptions that the cosine is unity, the sine is approximated by the angle, and products of small angles are to be neglected.)

The fundamental error analysis problem is that of determining ε_1, ε_2, ε_3 in terms of the errors—as yet unspecified. The next section discusses the problem of identifying the errors contributing to the perturbation of coordinate system 4.

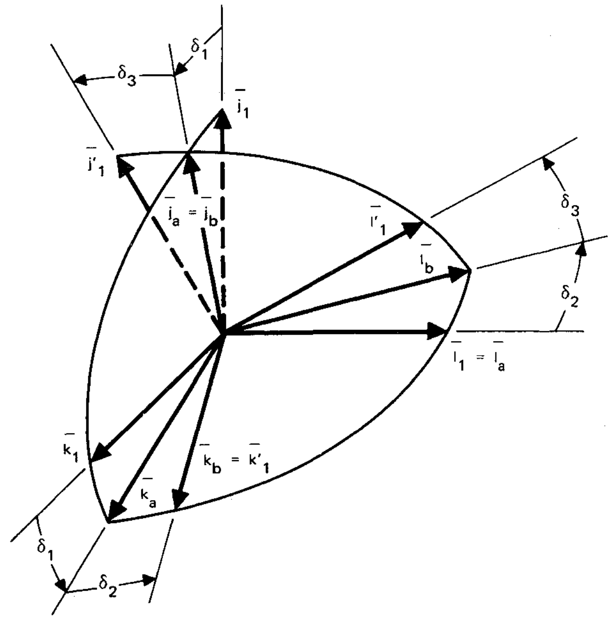

Fig. 5 Orientation of $\mathbf{k}_1'$, axis of rotation for α.

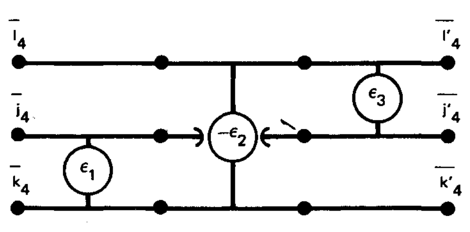

Fig. 4 Orientation of perturbed coordinate system.

Identification of Errors

It may appear that twelve unit vectors are required to define the four coordinate systems associated with a three-angle Euler sequence. However, three of the twelve unit vectors serve as axes for the Euler angle rotations and have duplicate names in each of two coordinate systems. Deletion of this duplication leaves nine distinct unit vectors, rather than twelve, associated with any three-angle sequence. Error analysis of a three Euler angle sequence requires that an angular error be associated with each

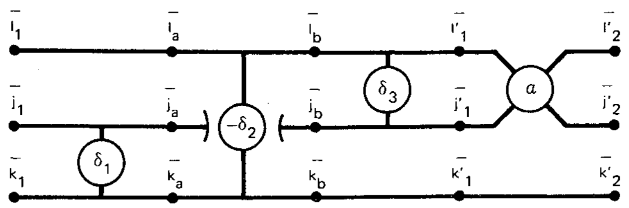

Fig. 6 Angular errors in first Euler rotation.

of these nine unit vectors. For the general case of a sequence of n finite Euler transformations the number of errors N that must be considered is given by

$$N = 3+2n \tag{1}$$

As an example consider the errors that must be considered in connection with the first rotation, α, in the sample problem. The unit vector $\mathbf{k}_1$, normal to the $\mathbf{i}_1$, $\mathbf{j}_1$ plane, is the nominal axis of rotation. The actual axis of rotation, $\mathbf{k}_1'$, will deviate from normality with the $\mathbf{i}_1$, $\mathbf{j}_1$ plane. In general, two angles are required to specify the orientation of a line relative to an orthogonal reference system. These two angles are δ_1 and δ_2 defined as infinitesimal rotations about the $\mathbf{i}_1$ and $\mathbf{j}_a$ unit vectors, respectively (Fig. 5). The Euler rotation angle α will itself be in error by an infinitesimal angle δ_3. The nominal axis of rotation for each of these three errors is easily identified in the Piogram for the first Euler angle rotation from Fig. 2. A Piogram can now be constructed showing the sequence of three infinitesimal rotations followed by the finite rotation α. The ordering of the three errors is not significant since second-order effects are assumed to be negligible. As a result the errors may be thought of as rotations nominally about one of the axes of the original orthogonal coordinate system $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$. Together the three infinitesimal rotations specify the orientation of the perturbed orthogonal coordinate system $\mathbf{i}_1'$, $\mathbf{j}_1'$, $\mathbf{k}_1'$. The Euler rotation α takes place relative to this perturbed coordinate system (Fig. 6). The intermediate coordinate system identified by subscripts a and b are of little significance as long as the errors are infinitesimal.

The remaining errors are introduced in a similar way. Unit vector $\mathbf{j}_2$, which serves as the axis of rotation for the next Euler angle β, is perturbed from its nominal orientation. As with the axis of rotation of α two error angles are required to specify the orientation of $\mathbf{j}_2'$. These are δ_3 and δ_4—rotations nominally about $\mathbf{k}_2$ and $\mathbf{j}_2$, respectively. Notice that the error δ_3 serves a dual purpose. It represents the error in the Euler angle α and also one of the errors in the orientation of the axis of rotation for the following Euler rotation. The angles δ_3 and α have the same axis of rotation and thus their order may be reversed or the two may be added to give α'. In Fig. 6 δ_3 could just as well have been shown following α and would perhaps more clearly show that δ_3 serves to specify one error in orientation of the axis of rotation for β and δ_4 the other.

Similarly, δ_5 is an error that represents the error in the Euler angle β as well as one error in the orientation of the axis, $\mathbf{i}_3'$, of the third Euler angle γ. δ_6 represents the second error required to specify the orientation of $\mathbf{i}_3'$. δ_7 represents the error in γ. Finally the errors δ_8 and δ_9 must be included to represent possible errors in the orientation of $\mathbf{i}_4$, $\mathbf{j}_4$, $\mathbf{k}_4$ relative to their nominal position. Although nine independent errors are sufficient to specify the error sources in the analysis, notice that each of the nine errors may have several individual contributors. For example, contributions to a single δ_i may arise from misalignments, bearing errors, nonorthogonality of axes, or inaccuracies in servos or transducers.

A Piogram can now be constructed that includes the angular errors and that is equivalent to Fig. 4 in the sense that it also defines a transformation from coordinate system 4 to coordinate system 4'. The diagram (Fig. 7) begins on the left with the inverse of the example three-angle sequence. This three-angle sequence is followed by a nine-angle sequence made up of the nine angular errors and three original angles. Three of the errors, δ_3, δ_5, δ_7, have been added to the corresponding Euler angles.

Notice that circled numbers in Figs. 1 and 2 identify the axes of rotation for the angular errors and that the number corresponds with the subscript on the associated error. The Piogram in Fig. 2 offers a convenient structure for assigning indices on the nine angular errors. Moving from left to right across the diagram, number the distinct unit vectors from top to bottom at each orthogonal set of unit vectors. On those unit vectors serving as axes for the Euler rotations, place the number at the midpoint. This process will always result in the errors δ_3, δ_5, δ_7 being associated with the Euler angles for all of the twelve possible Euler sequences.

The perturbed values of α, β, γ are α', β', γ' and are defined by

$$\alpha' = \alpha+\delta_3, \quad \beta' = \beta+\delta_5, \quad \gamma' = \gamma+\delta_7 \tag{2}$$

One of the particularly useful properties of the Piogram is the ease with which the scalar product of two unit vectors can be determined simply by writing the product of trigonometric functions contained in each path linking the two vectors, taking care to observe the correct signs. A path starts at the vector in the scalar product farthest to the left in the diagram and proceeds to the right ending at the second vector. The number of paths, and therefore the number of terms in the scalar product, can be shown to be a member of a Fibonacci sequence generated by the rule given by the last entry in the list of Piogram characteristics given previously.

For example the following scalar products can be written from Fig. 2 using the characteristics and sign convention implied by Fig. 3b.

$$\mathbf{j}_1\cdot\mathbf{i}_2 = \sin\alpha, \quad \mathbf{j}_1\cdot\mathbf{k}_3 = \sin\alpha\sin\beta$$
$$\mathbf{j}_1\cdot\mathbf{k}_4 = \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma$$

Notice that one path connects the vectors in $\mathbf{j}_1\cdot\mathbf{i}_2$ and $\mathbf{j}_1\cdot\mathbf{k}_3$ and therefore each contains only one term. Two paths connect the vectors in $\mathbf{j}_1\cdot\mathbf{k}_4$ and this scalar product therefore contains two terms.

The equivalence of Figs. 4 and 7, together with the method for determining scalar products from Piograms discussed above, provides an obvious solution to the example problem. Scalar products defining ε_1, ε_2, ε_3 may be identified directly from Fig. 4 (neglecting higher-order terms involving products of the δ_i's).

$$\begin{aligned} \varepsilon_1 &= \mathbf{k}_4\cdot\mathbf{j}_4' = -\mathbf{j}_4\cdot\mathbf{k}_4' \\ \varepsilon_2 &= \mathbf{i}_4\cdot\mathbf{k}_4' = -\mathbf{k}_4\cdot\mathbf{i}_4' \\ \varepsilon_3 &= -\mathbf{i}_4\cdot\mathbf{j}_4' = \mathbf{j}_4\cdot\mathbf{i}_4' \end{aligned} \tag{3}$$

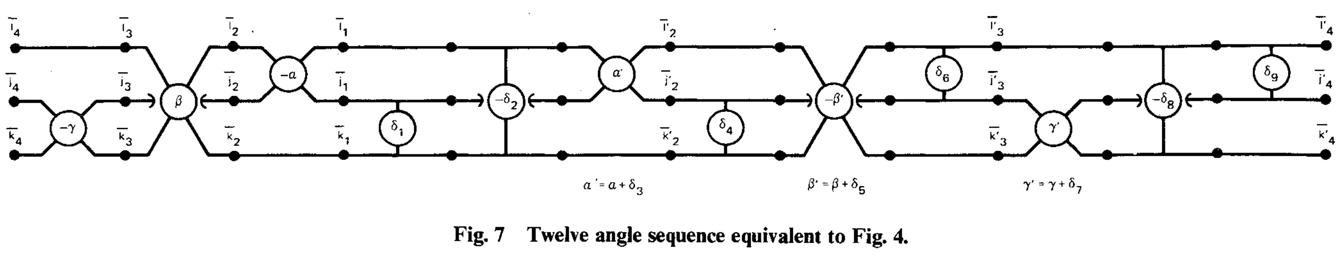

Fig. 7 Twelve angle sequence equivalent to Fig. 4.

In principle, these scalar products can be evaluated from Fig. 7 giving expressions for ε_1, ε_2, ε_3 in terms of α, β, γ and the δ_i's. This is equivalent to the traditional approach to the problem. Examination of Fig. 7 indicates how tedious this approach can be. There are 144 unique paths between $\mathbf{k}_4$ and $\mathbf{j}_4'$; thus the scalar product $\mathbf{k}_4 \cdot \mathbf{j}_4'$ consists of 144 terms. This result would be simplified by discarding terms involving products of the errors but a large effort is still involved in obtaining the answer by this approach.

A matrix equivalent to the solution given in Eq. (3) is developed in the next section and it is shown that a vector solution can be obtained by using similarity transformations. The vector solution gives a much more direct approach to the desired solution.

Solution by Matrix Techniques

A one-to-one correspondence (isomorphism) exists between the Piogram representation of a coordinate transformation and the equivalent matrix representation. Application of this isomorphic property to the Piogram showing the transformation from coordinate system 4 to 4′ (Fig. 4) gives directly the equivalent matrix representation (second-order terms are neglected).

$$\begin{bmatrix} \mathbf{i}_4' \\ \mathbf{j}_4' \\ \mathbf{k}_4' \end{bmatrix} = \begin{bmatrix} 1 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 1 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_4 \\ \mathbf{j}_4 \\ \mathbf{k}_4 \end{bmatrix} \tag{4}$$

The transformation matrix, $T_{44'}$, which accomplishes the transformation from coordinate system 4 to 4′ in Eq. (4) has the form

$$T_{44'} = \begin{bmatrix} 1 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 1 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 1 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{bmatrix} \tag{6}$$

$$= I + E \tag{7}$$

where I is the identity matrix and E is the skew-symmetric or antisymmetric error matrix with elements $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

The transformation matrix, $T_{44'}$, can be written in terms of α, β, γ and the δ_i's by applying the isomorphism between Piograms and matrices to Fig. 7

$$T_{44'} = [I + D_4] T_{34} [I + D_3] T_{23} [I + D_2] T_{12} [I + D_1] T_{21} T_{32} T_{43} \tag{8}$$

where D_1, D_2, D_3, D_4 are each skew-symmetric matrices defined in terms of the errors, δ_i

$$D_1 = \begin{bmatrix} 0 & \delta_3 & -\delta_2 \\ -\delta_3 & 0 & \delta_1 \\ \delta_2 & -\delta_1 & 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 & 0 & -\delta_5 \\ 0 & 0 & \delta_4 \\ \delta_5 & -\delta_4 & 0 \end{bmatrix} \quad D_3 = \begin{bmatrix} 0 & \delta_6 & 0 \\ -\delta_6 & 0 & \delta_7 \\ 0 & -\delta_7 & 0 \end{bmatrix} \quad D_4 = \begin{bmatrix} 0 & \delta_9 & -\delta_8 \\ -\delta_9 & 0 & 0 \\ \delta_8 & 0 & 0 \end{bmatrix} \tag{9}$$

and T_{12}, T_{23}, T_{34} are transformation matrices associated with the finite rotations, α, β, γ. Notice that the errors δ_3, δ_5, δ_7 are contained in the aforementioned skew-symmetric matrices and that the following matrices are in terms of the error-free Euler angles:

$$T_{12} = T_{21}{}^T = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_{23} = T_{32}{}^T = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \quad T_{34} = T_{43}{}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix} \tag{10}$$

The product of matrices in Eq. (8) can be rewritten as a sum of products, retaining only first-order error terms, by performing the indicated multiplications and neglecting the products of the infinitesimal skew-symmetric matrices containing the δ_i's. Thus,

$$T_{44'} = I + D_4 + T_{34} D_3 T_{43} + T_{34} T_{23} D_2 T_{32} T_{43} + T_{34} T_{23} T_{12} D_1 T_{21} T_{32} T_{43} \tag{11}$$

Equating $T_{44'}$, from Eqs. (7) and (11) gives

$$E = D_4 + T_{34} D_3 T_{43} + T_{34} T_{23} D_2 T_{32} T_{43} + T_{34} T_{23} T_{12} D_1 T_{21} T_{32} T_{43} \tag{12}$$

Equation (12) is a matrix equation. Expansion and algebraic manipulation of the right-hand side of Eq. (12) will yield, after considerable effort, expressions that can be equated with appropriate elements of the E matrix from the left-hand side of Eq. (12). Thus, explicit equations for the ε_i's in terms of α, β, γ and the δ_i's can, in principle, be obtained equivalent to those implied in Eq. (3).

The next section shows that properties of the similarity transformation can be applied to Eq. (12) giving a vector solution that is much more tractable than the matrix equation.

Similarity Transformation

The skew-symmetric matrix has some interesting characteristics that are useful in transforming Eq. (12) from a matrix equation to an equivalent vector equation.[3] The first of the characteristics is the property that a three-by-three skew-symmetric matrix can be put in a one-to-one correspondence (isomorphism) with a vector. That is, the matrix A with form

$$A = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \tag{13}$$

can be associated, under this isomorphism, with a vector

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \tag{14}$$

$$A \Leftrightarrow \mathbf{a} \tag{15}$$

The second characteristic of the skew-symmetric matrix useful in this analysis is the behavior of a skew-symmetric matrix under a similarity transformation. If A is a skew-symmetric matrix isomorphic with the vector $\mathbf{a}$ as in Eq. (15), then, if A is subjected to the following operation

$$B = TAT^T \tag{16}$$

where T is an orthogonal matrix and T^T its transpose, the matrix B will also be skew-symmetric. Furthermore, the vector $\mathbf{b}$, corresponding to B, is related to the vector $\mathbf{a}$, corresponding to A,

$$\mathbf{b} = T\mathbf{a} \tag{17}$$

Since the transformation matrices resulting from Euler angle transformations are orthogonal, each of the terms of Eq. (12) is of the form given in Eq. (16). Thus, the matrix equation in Eq. (12) can be transformed into the following vector equation by a term by term application of Eqs. (16) and (17).

$$\boldsymbol{\varepsilon} = \mathbf{d}_4 + T_{34}\mathbf{d}_3 + T_{34}T_{23}\mathbf{d}_2 + T_{34}T_{23}T_{12}\mathbf{d}_1 \tag{18}$$

where

$$\boldsymbol{\varepsilon} = \varepsilon_1 \mathbf{i}_4 + \varepsilon_2 \mathbf{j}_4 + \varepsilon_3 \mathbf{k}_4 \Leftrightarrow E \tag{19}$$

and

$$\begin{aligned} \mathbf{d}_1 &= \delta_1\mathbf{i}_1+\delta_2\mathbf{j}_1+\delta_3\mathbf{k}_1 \Leftrightarrow D_1 \\ \mathbf{d}_2 &= \delta_4\mathbf{i}_2+\delta_5\mathbf{j}_2 \Leftrightarrow D_2 \\ \mathbf{d}_3 &= \delta_6\mathbf{k}_3+\delta_7\mathbf{i}_3 \Leftrightarrow D_3 \\ \mathbf{d}_4 &= \delta_8\mathbf{j}_4+\delta_9\mathbf{k}_4 \Leftrightarrow D_4 \end{aligned} \tag{20}$$

The significance of the result expressed in Eq. (18) is simply that it is a vector equation and thus easier to work with than the equivalent matrix equation in Eq. (12).

Solution by Vector Techniques

The previous sections have shown that a vector solution to the error analysis problem can be obtained from the traditional matrix approach by application of the properties of skew-symmetric matrices under similarity transformations. It will now be shown that the vector solution can be obtained directly and that the direct vector approach suggests using the Piogram technique to obtain the solution with particular ease.

The relationships of Eq. (20) show that the error sources, δ_i, originally treated as infinitesimal rotations, can be equally well considered as vectors. Goldstein[4] presents a thorough and elegant discussion of the well-known isomorphism between infinitesimal rotations and axial vectors. This property suggests that the total error, $\boldsymbol{\delta}$, be written directly as a vector sum of the nine individual errors

$$\boldsymbol{\delta} = \delta_1\mathbf{i}_1+\delta_2\mathbf{j}_1+\delta_3\mathbf{k}_1+\delta_4\mathbf{i}_2+\delta_5\mathbf{j}_2+\delta_6\mathbf{k}_3+\delta_7\mathbf{i}_3+\delta_8\mathbf{j}_4+\delta_9\mathbf{k}_4 \tag{21}$$

or equivalently in terms of the ε_i which are to be determined

$$\boldsymbol{\delta} = \varepsilon_1\mathbf{i}_4+\varepsilon_2\mathbf{j}_4+\varepsilon_3\mathbf{k}_4 = \boldsymbol{\varepsilon} \tag{22}$$

Notice that in Eqs. (21) and (22) the δ_i and ε_i are written in terms of the nominal (i.e., unperturbed) coordinate axes. Thus again second-order terms are neglected. A solution of the error analysis problem is immediately evident from Eqs. (21) and (22). Thus,

$$\begin{aligned} \varepsilon_1 &= \boldsymbol{\delta}\cdot\mathbf{i}_4 \\ \varepsilon_2 &= \boldsymbol{\delta}\cdot\mathbf{j}_4 \\ \varepsilon_3 &= \boldsymbol{\delta}\cdot\mathbf{k}_4 \end{aligned} \tag{23}$$

using $\boldsymbol{\delta}$ from Eq. (21).

Solution by Piogram Technique

Two equivalent vector solutions to the sample error analysis problem have been obtained in the preceding analysis and are repeated here. By matrix techniques:

$$\boldsymbol{\varepsilon} = \mathbf{d}_4+T_{34}\mathbf{d}_3+T_{34}T_{23}\mathbf{d}_2+T_{34}T_{23}T_{12}\mathbf{d}_1 \tag{18}$$

By vector techniques:

$$\begin{aligned} \varepsilon_1 &= \boldsymbol{\delta}\cdot\mathbf{i}_4 \\ \varepsilon_2 &= \boldsymbol{\delta}\cdot\mathbf{j}_4 \\ \varepsilon_3 &= \boldsymbol{\delta}\cdot\mathbf{k}_4 \end{aligned} \tag{23}$$

where $\boldsymbol{\delta}$ in Eq. (23) is given by Eq. (21).

Both of these equivalent formulations call for the resolution of each of the error sources, δ_i, into coordinate system 4 where they add to form the total errors, ε_i.

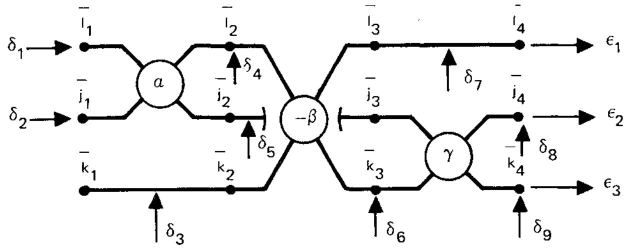

Fig. 8 Solution of sample problem.

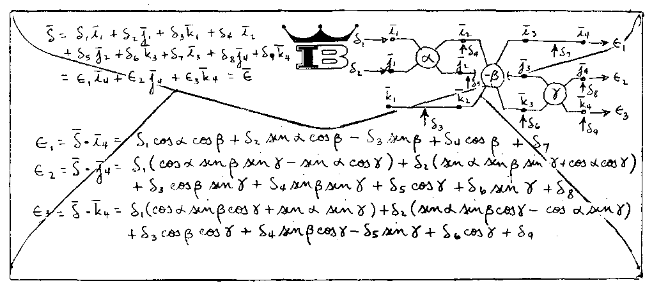

Fig. 9 Complete solution of example problem.

The coordinate transformations implicitly called for in Eq. (18) or (23) can be easily and directly obtained from a Piogram (Fig. 8). Notice the correspondence between Figs. 2 and 8. It is apparent that construction of Fig. 2 is, in fact, a sufficient basis for the solution of the example problem. Thus, from either Fig. 2 or 8, the explicit solution for the ε_i in terms of the δ_i and α, β, γ can be directly written

$$\begin{aligned} \varepsilon_1 &= \delta_1\cos\alpha\cos\beta+\delta_2\sin\alpha\cos\beta-\delta_3\sin\beta+\delta_4\cos\beta+\delta_7 \\ \varepsilon_2 &= \delta_1(\cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\gamma)+ \\ &\quad \delta_2(\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma)+\delta_3\cos\beta\sin\gamma+ \\ &\quad \delta_4\sin\beta\sin\gamma+\delta_5\cos\gamma+\delta_6\sin\gamma+\delta_8 \\ \varepsilon_3 &= \delta_1(\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma)+ \\ &\quad \delta_2(\sin\alpha\sin\beta\cos\gamma-\cos\alpha\sin\gamma)+\delta_3\cos\beta\cos\gamma+ \\ &\quad \delta_4\sin\beta\cos\gamma-\delta_5\sin\gamma+\delta_6\cos\gamma+\delta_9 \end{aligned} \tag{24}$$

These equations can be used to determine the sensitivity of one of the ε_i to a particular δ_i. In a particular case one or more of the δ_i may be negligible. In many problems all the errors except those in the angles of rotation δ_3, δ_5, δ_7 are negligible. The inverse of the isomorphism used to obtain the ε_i as components of a vector may be invoked to obtain the transformation matrix, Eq. (5) or the equivalent Piogram in Fig. 4, which relates coordinate systems 4 and 4′.

Conclusion

An example problem has been posed and solved to illustrate various approaches to the solution of the error analysis problem for Euler angle transformations. The techniques used for solution of the example problem are applicable to any other three-angle sequence and, by an obvious extension, to sequences of angular rotations of any length. The principal results of the paper are 1) the presentation of an orderly, unambiguous way to determine the number of error sources and their representation as vector quantities; 2) a demonstration that the vector solution can be obtained in a simple direct way; and 3) a demonstration that the Piogram technique offers a particularly clear and compact basis for formulating and solving the Euler angle error analysis problem. The significance of these results is that they show that error analysis of any Euler angle sequence by the recommended technique is an almost trivial task as compared with the usual tedious approach by matrix techniques. The problem and its solution may literally be exhibited on the back of an envelope (Fig. 9).

References

[1] Pio, R. L., "Symbolic Representation of Coordinate Transformations," *IEEE Transactions on Aerospace and Navigational Electronics*, Vol. ANE-11, June 1964, pp. 128–134.

[2] Pio, R. L., "Euler Angle Transformations," *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 4, Oct. 1966, pp. 707–715.

[3] Broxmeyer, C., *Inertial Navigation Systems*, McGraw-Hill, New York, 1964, p. 28.

[4] Goldstein, Herbert, *Classical Mechanics*, Addison-Wesley, Reading, Mass., 1959, pp. 124–132.